

Fermions on random lattices

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We put fermions and define the Dirac operator and spin structures on a randomly triangulated 2d manifold.

1. Introduction

It is known, that for more than 2d the constructive, lattice approach towards gravity quantization faces the problem of the apparent instability of random manifolds. Several ideas are under discussion. One is that a sensible quantum gravity theory should be supersymmetric in the continuum limit. World-sheet supersymmetry is necessarily broken on the lattice. But, perhaps, it is sufficiently weakly broken to stabilize the manifolds. The problem is notoriously very difficult. The first step in this direction is to define a spin structure on a piecewise linear, randomly triangulated manifold. We consider here 2d manifolds made up of equilateral triangles. Generalizations of our method to arbitrary triangulations and to $d > 2$ are possible, but beyond the scope of this report. For more details see [1].

2. Spin connection and spin structures

We consider a randomly triangulated manifold. The fermions are assumed to live on the dual lattice. Our starting point is the familiar gauged Wilson action for fermions:

$$S = \sum_{\langle b,a \rangle} \bar{\psi}(b) [-\kappa(1 + n_{ba} \cdot \gamma) U(ba) + \frac{1}{4} \delta_{ba}] \psi(a) \quad (1)$$

Here b, a are two neighboring sites (triangles), n_{ba} is a unit vector along the oriented link ba and κ is the hopping parameter. U_{ba} is a connection matrix which transports the spinor ψ from a to b . In the standard lattice gauge theory U_{ba} operates in the internal symmetry space. In quantum gravity it connects the neighbor local frames.

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Let us recall, that the group of general coordinate transformations has no spinor representation. Thus, in general relativity, in order to define spinors one has to introduce local orthonormal frames transforming under $SO(d)$. In our case, the manifold is piecewise linear, and a local frame $\{e^j(a)\}$ is associated with each triangle a . The parallel transport from a to b along some curve C is:

$$e^j(b) = U(b \xleftarrow{C} a)_k^j e^k(a) \quad (2)$$

This is to be lifted to the spinor representation

$$\psi^\alpha(b) = \mathcal{U}(b \xleftarrow{C} a)_\beta^\alpha \psi^\beta(a) \quad , \quad \mathcal{U} = [U]_{1/2} \quad (3)$$

Here one encounters the well-known sign ambiguity: the matrix \mathcal{U} is defined up to sign. However, when C is a loop, the sign of $\mathcal{U}(a \xleftarrow{C} a)$ is determined by the sign of any other loop into which C can be continuously deformed. Consequently, when C is contractable to a point, like every loop on a sphere, the sign of $\mathcal{U}(a \xleftarrow{C} a)$ is actually determined and a unique spin structure can be defined. For non-contractable loops there are two possible sign choices. Since there are two types of non-contractable loops on a torus, four distinct spin structures can be defined on it. This generalizes in an obvious way to orientable surfaces of higher genus. On the Klein bottle, however, one meets an obstruction: the spin structure cannot be defined. The problem of putting fermions on a random lattice boils down to that of defining the spin structure on it consistently.

3. The explicit construction of the connection

The local frames in two neighboring triangles

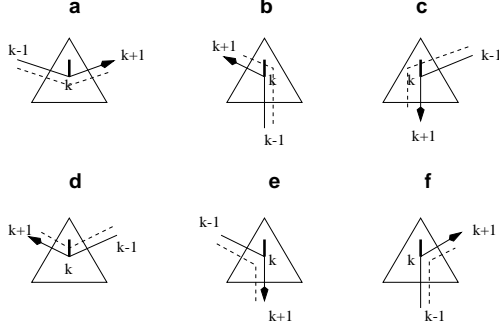


Figure 1. Six possible ways a dual lattice loop can go through a triangle. The line segment pointing up from the centre of the triangle is the vertex flag indicating the gauge choice, i.e. the direction of the vielbein $e^1(k)$. The dashed line is the loop slightly displaced to the right. The sign factor in Table 1 is negative when the dashed line crosses the flag.

are connected by $e(b) = U(ba)e(a)$. The matrix notation is used to drop the indices appearing in (2). To the local gauge transformation of the frames $e(a) \rightarrow \Omega(a)e(a)$ corresponds the transformation $U(ba) \rightarrow \Omega(b)U(ba)\Omega^{-1}(a)$. Clearly, $TrU(C)$ is gauge invariant if C is a closed loop.

Let $R(\phi) = \exp(\epsilon\phi)$ denote the matrix performing the rotation by angle ϕ (ϵ is the rotation generator). It is easy to convince oneself that $U(ba)$ can be written

$$U(ba) = R^{-1}(\phi_{b \rightarrow a})R(\pi)R(\phi_{a \rightarrow b}) \quad (4)$$

where $\phi_{a \rightarrow b}$ is the angle between the 1st axis of $e(a)$ and n_{ab} . Likewise, $\phi_{b \rightarrow a}$ is the axis of the 1st axis of $e(b)$ and n_{ba} . The angles are oriented: they are always measured counterclockwise from e to the appropriate n . The parallel transporter along a closed loop is the product of successive rotations $R(\pi)R(\phi_{k \rightarrow k-1})R^{-1}(\phi_{k \rightarrow k+1}) = R(\pm\pi/3)$, the sign \pm depending on whether the loop turns right or left. For an elementary loop L_q , encircling a single vertex of the triangulated lattice in q steps, one obviously has

$$\frac{1}{2}TrU(L_q) = \frac{1}{2}TrR(\pm\pi/3) = \cos(q\pi/3) \quad (5)$$

As expected the rhs of (5) equals unity when the surface is flat, i.e. when $q = 6$ (remember that the triangles are equilateral!). For general q the rhs equals the cosine of the monodromy angle and measures the curvature.

In the spinor representation we write in analogy to (4):

$$\mathcal{U}(ba) = g_{ba}\mathcal{R}^{-1}(\phi_{b \rightarrow a})\mathcal{R}(\pi)\mathcal{R}(\phi_{a \rightarrow b}) \quad (6)$$

where $\mathcal{R}(\phi) = \exp(\epsilon\phi/2)$ (in 2d the rotation generator in both representations is the same matrix) and g_{ba} is a sign factor. Our problem, actually a topological problem, is to fix the sign factors g_{ba} all over the lattice for a given gauge. Of course, physical quantities are gauge independent.

The spinor analogue of (5) is

$$\frac{1}{2}Tr\mathcal{U}(L_q) = F(L_q)\cos(q\pi/6) \quad (7)$$

where $F(L_q)$ is a product of g 's appearing in (6) and of additional sign factors, to be calculated in a moment and given in Table 1. Clearly, for $q = 6$ the rhs must equal unity and therefore $F(L_6) = -1$. One can produce several argument to show that

$$F(L_q) = -1 \quad \forall q \quad (8)$$

E.g. one can smear the metric singularity at the vertex within L_q . For an infinitesimal loop the deficit angle is then zero and the loop sign factor is -1. As one enlarges the loop, the deficit angle is progressively built. The loop sign factor keeps its value -1, since $Tr\mathcal{U}(L_q)$ varies continuously. Eq. (8) is an essential constraint.

The calculation of the rhs of (7) is similar to that leading to (5). Again the loop turns by $\pm\pi/3$ modulo 2π , but now the 2π is not innocent, since in the spinor representation it is the half-angle that matters. This can possibly yield an extra sign factor, which depends on how the dual loop goes through the successive triangles, say $k-1, k$ and $k+1$. It is convenient to choose the gauge where $e^1(k)$ points from the center of the triangle k towards one of its vertices. The six possible cases are shown in Fig. 1. The relevant result is given in Table 1. The trace of the product of the Kac-Ward factors $\exp(\pm\epsilon\pi/6)$ yields $\cos(q\pi/6)$. The loop factor $F(L_q)$ is a product of g 's and of

Table 1

The last column is $\mathcal{R}(\pi)\mathcal{R}(\phi_{k \rightarrow k-1})\mathcal{R}^{-1}(\phi_{k \rightarrow k+1})$ and is a product of a specific sign factor and of an exponential, the Kac-Ward factor.

Fig 1	$\phi_{k \rightarrow k-1}$	$\phi_{k \rightarrow k+1}$	sign \times Kac-Ward
a	$\pi/3$	$5\pi/3$	$+\exp(-\epsilon\pi/6)$
b	π	$\pi/3$	$-\exp(-\epsilon\pi/6)$
c	$5\pi/3$	π	$-\exp(-\epsilon\pi/6)$
d	$5\pi/3$	$\pi/3$	$-\exp(+\epsilon\pi/6)$
e	$\pi/3$	π	$+\exp(+\epsilon\pi/6)$
f	π	$5\pi/3$	$+\exp(+\epsilon\pi/6)$

the sign factors to be read from Table 1. The global gauge choice can be represented graphically, by attaching flags to dual vertices and links. A vertex flag is just the direction of e^1 (one can turn it to the right or to the left, provided one does not cross any dual link). A link flag is put on the right of the link ba if $g_{ba} = -1$. It follows, that the topological constraint (8) is satisfied if the number of flags inside every elementary loop is odd.

4. Loop signs and topology

The flags are a useful tool, which helps proving the following two theorems:

- **T1** : For an arbitrary orientable triangulation one can choose the gauge, i.e. the orientations of the local frames and the link sign factors, so that $F(L) = -1$ for every elementary loop L .

The idea of the proof consists in checking first the validity of T1 for a minimal sphere. Then one verifies that an ergodic triangulation building algorithm is compatible with the theorem. Finally, by gluing spheres in an appropriate manner one extends the result to a sphere with handles.

- **T2** : One has $F(C) = -1$ for every contractable loop.

This can be checked by gluing elementary loops.

5. Wilson fermions on a randomly triangulated manifold

One has

$$\gamma_{ba} \equiv n_{ba} \cdot \gamma = \mathcal{R}^{-1}(\phi_{b \rightarrow a})\gamma^1\mathcal{R}(\phi_{b \rightarrow a}) \quad (9)$$

According to (1) the mass independent part of the Dirac operator is

$$D(ba) = \frac{1}{2}(1 + \gamma_{ba})\mathcal{U}_{ba} \quad (10)$$

Hence, finally

$$D(ba) = g_{ba}\mathcal{R}^{-1}(\phi_{b \rightarrow a})\frac{1}{2}(1 + \gamma^1)\mathcal{R}(\phi_{a \rightarrow b}) \quad (11)$$

The Dirac operator is constructed once all the flags are put. This determines the angles and the g 's. With our choice of gauge the angles ϕ take the values $\pi/3, \pi, 5\pi/3$. Thus, there are nine possibilities for the matrix on the rhs of (11). These nine matrices can be calculated beforehand.

Using (11) one readily calculates the Majorana fermion loop expectation value

$$\langle \bar{\psi}(1)D(12)\dots D(n1)\psi(1) \rangle = -F(C)(\sqrt{3}/2)^n \quad (12)$$

When the loop is contractable $F(C) = -1$ and the rhs is just $(\sqrt{3}/2)^n$. Since there is a one-to-one correspondence between contractable loops and Ising spin domain boundaries, this result can be used [2] to demonstrate the equivalence of Ising spins and Majorana fermions on a sphere. The result can be extended to arbitrary orientable surfaces, provided in the fermion theory one sums over all possible spin structures. Only then the contribution of unpaired non-contractable fermion loops cancels. The similarity between this prescription and the GSO projection has been noted by Polyakov [3].

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